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ON RELIABLE CONTROL SYSTEM DESIGNS WITH AND WITHOUT FEEDBACK RECONFIGURATIONS *

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Abstract

This paper contains an overview of a theoretical framework for the design of reliable multivariable control systems, with special emphasis on actuator failures and necessary actuator redundancy levels. Using a linear model of the system, with Markovian failure probabilities and quadratic performance index, an optimal stochastic control problem is posed and solved. The solution requires the iteration of a set of highly coupled Riccati-like matrix difference equations; if these converge one has a reliable design; if they diverge, the design is unreliable, and the system design cannot be stabilized.

In addition, it is shown that the existence of a stabilizing constant feedback gain and the reliability of its implementation is equivalent to the convergence properties of a set of coupled Riccati-like matrix difference equations.

In summary, these results can be used for off-line studies relating the open loop dynamics, required performance, actuator mean time to failure, and functional or identical actuator redundancy, with and without feedback gain reconfiguration strategies.

1. Introduction

This paper is an overview of a research effort which addresses some of the current problems in interfacing systems theory and reliability, and puts this research in perspective with the open questions in this field. Reliability is a relative concept; it is, roughly, the probability that a system will perform according to specifications for a given amount of time. The motivating question behind this report is: What constitutes a reliable system?

If a theory were available which allowed a comparison between alternate designs, based on both the expected system reliability and the expected system performance, it would greatly simplify the current design methodology. It is unfortunate that at present there is no accepted methodology for a determination of expected system performance which accounts for changes in the performance characteristics due to failure, repair or reconfiguration of system functions. This report presents such a methodology for a specific class of linear systems with quadratic cost criteria.

Basically, the reliability of a system is the probability that the system will perform according to specifications for a given amount of time. In a system-theoretic context, the specification which a system must meet is stability; also, since, at least for most mathematical models of systems, stability is a long-term attribute of the system, the amount of time for which the system must remain stable is taken to be infinite. Therefore, the following definitions of system reliability are used in this paper:

Definition 1: A system (implying the hardware configuration, or mathematical model of that configuration, and its associated control and estimation structure) has reliability r where r is the probability that the system will be stable for all time.

Definition 2: A system is said to be reliable if $r = 1$.

Definition 3: A system design, or configuration, is reliable if it is stabilizable with probability one.

These definitions of reliability depend on the definition of stability, and for systems which can have more than one mode of operation, stability is not that easy to determine. In this paper, stability will mean either mean-square stability (over some random space which will be left unspecified for the moment), or cost-stability (again, an expectation over a certain random space), which is basically the property that the accumulated cost of system operation is bounded with probability one. (The definition of cost is also deferred.)

The reliability of a system will depend on the reliabilities of its various components and on their interconnections. Thus, the systems engineer must have an understanding of the probabilistic mechanisms of component failure, repair, and system reconfiguration.

Component failures, repairs, and reconfigurations are modeled in this paper by a Markov chain. Only catastrophic changes in the system structure are considered; degradations are not modeled. The hazard rate is assumed to be constant, resulting in an exponential failure distribution. In the discrete-time case, to which this paper is confined exclusively, the hazard rate becomes the probability of failure (or repair or reconfiguration) between time t and time $t+1$.

It is now necessary to define precisely the modes of operation and their dynamic transitions. The terms system configuration and system structure will be used.

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Definition 4: System Structure: A possible mode of operation for a given system; the components, their interconnections, and the information flow in the system at a given time.

Definition 5: System Configuration: The original design of the system, accounting for all modeled modes of operation, and the Markov chain governing the configuration, or structural, dynamics (transitions among the various structures). In this paper, structures are referenced by convention by the set of non-negative integers

$$I \in \{0,1,2,3,\ldots,L\} \tag{1.1}$$

An important question in reliability is the effect of redundancy on system performance. In other words, how should the allocation of control resources be allocated to the redundant components, and how should the component reliabilities affect the choice of an optimal control law? The control methodologies presented in this paper answer the question for a specific class system configurations. They yield a quantitative analysis of the effectiveness of a given system design, where effectiveness is a quantity relating both the performance and the reliability of a configuration design.

Previously, several authors have studied the optimal control of systems with randomly varying structure. Most notable among these is Wonham [1], where the solution to the continuous time linear regulator problem with randomly jumping parameters is developed. This solution is similar to the discrete time switching gain solution presented in Section 3. Wonham also proves an existence result for the steady-state optimal solution to the control of systems with randomly varying structure; however, the conclusion is only sufficient; it is not necessary. Similar results were obtained in Beard [2] for the existence of a stabilizing gain, where the structures were of a highly specific form; these results were necessary and sufficient algebraic conditions, but cannot be readily generalized to less specific classes of systems. Additional work on the control problem for this class of systems has been done by Sworder [3], Rather & Luenberger [4], Bar-Whalom & Sivan [5], Willner [6] and Pierce & Sworder [7]. The dual problem of state estimation with a system with random parameter variations over a finite set was studied in Chang & Athans [8].

Recently, the robustness of the linear quadratic regulator has been studied by Wong, et. al. [9] and Safonov & Athans [10]. Section 6 of this paper gives necessary and sufficient conditions for the existence of a robust linear constant gain control law for a specific class of systems.

Some of the preliminary results on which this research was based were presented in unpublished form at the 1977 Joint Automatic Control Conference in San Francisco by Birdwell, and published for the 1977 IEEE Conference on Decision and Control Theory in New Orleans by Birdwell & Athans [11]. This paper is based on the results in Birdwell [12].

There are two major contributions of this research. First, the classification of a system design as reliable or unreliable has been equated with the existence of a steady-state switching gain and cost for that design. If this gain does not exist, then the system design cannot be stabilized; hence, it is unreliable. The only recourse in such a case is to use more reliable components and/or more redundancy. Reliability of a system design can therefore be determined by a test for c convergence of the set of coupled Riccati-like equation as the final time goes to infinity.

The second major contribution lies in the robustness implications. Precisely, a constant gain for a linear feedback control law for a set of linear systems is said to be robust if that gain stabilizes each linear system individually, i.e., without regard to the configuration dynamics. The problem of determining when such a gain exists, and of finding a robust gain, can be formulated in the context of this research. As a result, this methodology gives an algorithm for determining a robust gain for a set of linear systems which is optimal with respect to a quadratic cost criterion. If the algorithm does not converge, then no robust gain exists.

For the purpose of brevity, most result will be stated without proof. The reader may find these proofs in reference [12], and in the papers currently in preparation.

2. Problem Statement

Consider the system

$$\underline{x}_{t+1} = \underline{A}\,\underline{x}_t + \underline{B}_{k(t)}\,\underline{u}_t \tag{2.1}$$

where

$$\underline{x}_t \in R^n \tag{2.2}$$

$$\underline{u}_t \in R^m \tag{2.3}$$

$$\underline{A} \in R^{n\times n} \tag{2.4}$$

and, for each k, an element of an indexing set I

$$k \in I = \{0,1,2,\ldots,L\} \tag{2.5}$$

$$\underline{B}_k \in R^{n\times m} \tag{2.6}$$

where

$$\underline{B}_k \in \{\underline{B}_i\}_{i\in I} \tag{2.7}$$

The index k(t) is a random variable taking values in I which is governed by a Markov chain and

$$\underline{\pi}_{t+1} = \underline{P}\,\underline{\pi}_t \tag{2.8}$$

$$\underline{\pi}_t \in R^{L+1} \tag{2.9}$$

where $\pi_{i,t}$ is the probability of $k(t) = i$, given no on-line information about k(t), and $\underline{\pi}_0$ is the initial distribution over I.

It is assumed that the following sequence of events occurs at each time t:

1) $\underline{x}_t$ is observed exactly
2) then $\underline{B}_{k(t-1)}$ switches to $\underline{B}_{k(t)}$
3) then $\underline{u}_t$ is applied.

Consider the structure set $\{\underline{B}_k\}_{k\in I}$ indexed by I. Define the structural trajectory $\overline{X}_T$ to be a sequence of elements k(t) in I which select a specific structure $\underline{B}_{k(t)}$ at time t,

$$\overline{X}_T = (k(0),k(1),\ldots,k(T-1)) \tag{2.10}$$

The structural trajectory $\overline{X}_T$ is a random variable with probability of occurance generated from the Markov equation (2.8).

$$p(\bar{x}_T) = \prod_{t=0}^{T-1} \pi_{k(t),t} \qquad (2.11)$$

where the control interval is

$$\{0,1,2,\ldots,T-1,T\} \qquad (2.12)$$

for the finite time problem with terminal time T. Then for a given state and control trajectory

$(\underline{x}_t, \underline{u}_t)_{t=0}^{T-1}$ generated by (2.1) and $\bar{x}_T$ from a sequence of controls $(\underline{u}_t)_{t=0}^{T-1}$, the cost index is to be the standard quadratic cost criterion

$$\mathcal{J}_T(\bar{x}_T, (\underline{x}_t, \underline{u}_t)_{t=0}^{T-1}) = \sum_{t=0}^{T-1} \underline{x}_t^T \underline{Q} \underline{x}_t + \underline{u}_t^T \underline{R} \underline{u}_t + \underline{x}_T^T \underline{Q} \underline{x}_T \qquad (2.13)$$

The objective is to choose a feedback control law, which may depend on any past information about $\underline{x}_t$ or $\underline{u}_t$, mapping $\underline{x}_t$ into $\underline{u}_t$

$$\phi_t^* : R^n \rightarrow R^m \qquad (2.14)$$

$$\phi_t^* : \underline{x}_t \rightarrow \underline{u}_t \qquad (2.15)$$

such that the expected value of the cost function $\mathcal{J}_T$ from equation (2.13)

$$J_T = E[\mathcal{J}_T \mid \underline{\pi}_0] \qquad (2.16)$$

is minimized over all possible mappings $\underline{\phi}_t$ at $\underline{\phi}_t^*$.

3. The Optimal Solution

Normally, a control law of the form (2.15) must provide both a control and an estimation function in this type of problem; hence the label dual control is used. Here, the structure of the problem allows the exact determination of k(t-1) from $\underline{x}_t$, $\underline{x}_{t-1}$ for almost all values of $\underline{u}_{t-1}$. This result is stated in the following lemma

Lemma I [12]: For the set $\{\underline{B}_k\}_{k \in I}$, where the $\underline{B}_k$'s are distinct, the set $\{\underline{x}_{k,t+1} = \underline{A}\underline{x}_t + \underline{B}_k \underline{u}_t\}_{k=0}^{L}$ has distinct members for almost all values of $\underline{u}_t$.

Ignoring the set of controls of measure zero for which the members of

$$\{\underline{x}_{k,t+1}\}_{k=0}^{L} \qquad (3.1)$$

are not distinct, then for (almost) any control which the optimal algorithm selects, the resulting state $\underline{x}_{t+1}$ can be compared with the members of the set (3.1) for an exact match (of which there is only one with probability 1), and k(t) is identified as the generator of that matching member $\underline{x}_{k,t+1}$.

Since perfect identification is the best any algorithm can achieve, the optimal control law $\underline{u}_t^* = \underline{\phi}_t^*(\underline{x}_t)$ can be calculated with the assumption that k(t-1) is known, since this is the case with probability one. Thus, this solution will be labeled the switching gain solution, since, for each time t, L+1 optimal solutions are calculated apriori, and one solution is chosen on-line for each time t, based on the past measurements $\underline{x}_t$, $\underline{x}_{t-1}$ and $\underline{u}_t$, which yield perfect knowledge of k(t-1).

Dynamic programming is used to derive the optimal switching gain solution. It has been proved [12] that at each time t, the optimal expected cost-to-go, given the system structure k(t-1), is

$$V^*(\underline{x}_t, k(t-1), t) = \underline{x}_t^T \underline{S}_{k,t} \underline{x}_t \qquad (3.2)$$

where the $\underline{S}_{k,t}$ are determined by a set of L+1 coupled Riccati-like equations (one for each possible configuration):

$$\underline{S}_{k,t} = \underline{A}^T \left\{ \sum_{i=0}^{L} p_{ik} \underline{S}_{i,t+1} - \left[\sum_{i=0}^{L} p_{ik} \underline{S}_{i,t+1} \underline{B}_i \right] \left[\underline{R} + \sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \right]^{-1} \cdot \left[\sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \right] \right\} \underline{A} + \underline{Q} \qquad (3.3)$$

The optimal control, given k(t-1) = k, is

$$\underline{u}_{k,t}^* = -\left[\underline{R} + \sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \right]^{-1} \cdot \sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{A} \underline{x}_t \qquad (3.4)$$

Writing

$$\underline{u}_{k,t}^* = \underline{G}_{k,t} \underline{x}_t \qquad (3.5)$$

then

$$\underline{G}_{k,t} = -\left[\underline{R} + \sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \right]^{-1} \cdot \sum_{i=1}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{A} \qquad (3.6)$$

Thus, $\underline{u}_t^* = \phi_t(\underline{x}_t)$ is a switching gain linear control law which depends on k(t-1). The variable k(t-1) is determined by

$$k(t-1) = i \ \text{ iff } \underline{x}_t = \underline{A}\underline{x}_{t-1} + \underline{B}_i \underline{u}_{t-1} \qquad (3.7)$$

Note that the $\underline{S}_{i,t}$'s and the optimal gains $\underline{G}_{k,t}$ can be computed off-line and stored. Then at each time t, the proper gain is selected on-line from k(t-1), using equation (3.7), as in Figure 1.

This solution is quite complex relative to the structure of the usual linear quadratic solution. Each of the Riccati-like equations (3.7) involves the same complexity as the Riccati equation for the linear quadratic solution. In addition, there is the on-line complexity arising from the implementation of gain scheduling. In Section 4, a non-switching gain solution will be presented which has an identical on-line structure to that of the linear quadratic solution, but has similar off-line computational complexity to that of the switching gain solution. This non-switching gain solution is suboptimal, but requires less on-line complexity.

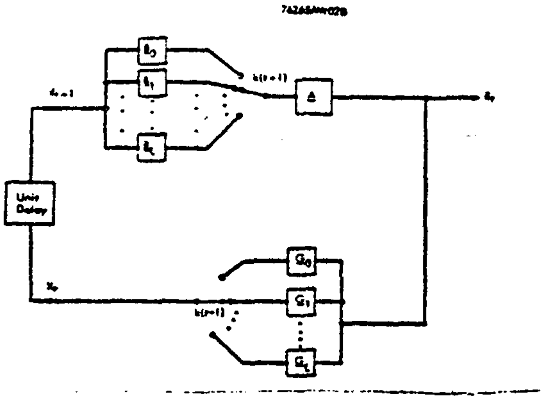

Figure 1: The switching gain control law.

A steady-state solution to equation (3.7) may exist, but the conditions for its existence are unknown. The steady-state solution would have the advantage that only one set of gains need be stored on-line, instead of requiring a set of gains to be stored for each time t. Since the steady-state solution is simply the value to which equation (3.7) converges as it is iterated backward in time, at present, the equations can be iterated numerically until either they converge or meet some test of non-convergence. The possibility of limit cycle solutions in the switching gain computations is excluded by the following lemma:

Lemma 2 [12]: If the optimal expected cost-to-go at time t is bounded for all t, then equation (3.7) converges.

The existence of a steady-state solution to the switching gain problem establishes a division of system designs into those which are inherently reliable and those which are unreliable. Even though conditions to test for the existence of the steady-state solution are unavailable, software can be used with iteration for the test.

4. The Non-switching Gain Solution

In this Section, the permissible controls are restricted to be of the linear feedback form

$$\underline{u}_t = \underline{G}_t \underline{x}_t \tag{4.1}$$

where the gain matrix G_t is restricted to be a function only of time and the initial conditions; i.e., it cannot depend on x_t or π_t. The objective is to minimize over the set of admissible controls the expectation of (2.13) where the expectation is taken over the set of possible structural trajectories

$$\tilde{x}_T \in \prod_T I \tag{4.2}$$

and the set of initial conditions $\underline{x}_0$. The resulting control law will be a non-switching law, using gains determined apriori.

Thus, the optimal control law $\underline{u}^*_t = \underline{G}^*_t \underline{x}_t$ should minimize the cost

$$\begin{aligned} \bar{J}_T &= E[\mathcal{J}_T | \Pi_0] \\ &= E\left[\sum_{t=0}^{T-1} \underline{x}_t^T Q \underline{x}_t + \underline{u}_t^T R \underline{u}_t + \underline{x}_T^T Q \underline{x}_T \,\Big|\, \Pi_0\right] \end{aligned} \tag{4.3}$$

over the set of admissible controls.

Since the structure of $\underline{u}_t = \underline{G}_t \underline{x}_t$ is fixed, the problem is equivalent to minimizing, in an open-loop sense, the cost function

$$\begin{aligned} E[\mathcal{J}_T | \Pi_0] = E\Bigg[&\sum_{t=0}^{T-1} \underline{x}_t^T Q \underline{x}_t + \underline{x}_t^T \underline{G}_t^T R \underline{G}_t \underline{x}_t \\ &+ \underline{x}_T^T Q \underline{x}_T \,\Big|\, \Pi_0 \Bigg] \end{aligned} \tag{4.4}$$

with respect to the gain matrix $\underline{G}_t$, $t=0,1,\ldots,T-1$.

The matrix minimum principle of Athans [13] is used to determine the necessary conditions for the existence of $\underline{u}^*_t$ (or equivalently, $\underline{G}^*_t$). Let the initial state $\underline{x}_0$ be a zero mean random variable which is independent of any structure. Let

$$\underline{\Sigma}_0 = E[\underline{x}_0 \underline{x}_0^T | \Pi_0] = E[\underline{x}_0 \underline{x}_0^T] \tag{4.5}$$

be the covariance matrix of $\underline{x}_0$. Defining the covariance of $\underline{x}_t$ as

$$\underline{\Sigma}_t \triangleq E[\underline{x}_t \underline{x}_t^T | \Pi_0] \tag{4.6}$$

and if we define

$$\underline{\Sigma}_{i,t} = E[\underline{x}_t \underline{x}_t^T | k(t-1)=i, \Pi_0] \tag{4.7}$$

then the matrix $\underline{\Sigma}_{i,t}$ can be defined recursively as

$$\underline{\Sigma}_{j,t+1} = \frac{1}{\pi_{j_t}} \sum_{i=0}^{L} p_{ji} \pi_{i_{t-1}} (\underline{A} + \underline{B}_j \underline{G}_t) \underline{\Sigma}_{i,t} (\underline{A} + \underline{B}_j \underline{G}_t)^T \quad \text{for } t > 1. \tag{4.8}$$

$$\underline{\Sigma}_{j,i} = (\underline{A} + \underline{B}_j \underline{G}_0) \underline{\Sigma}_0 (\underline{A} + \underline{B}_j \underline{G}_0)^T \tag{4.9}$$

and the relation

$$\underline{\Sigma}_t = \sum_{i=0}^{L} \pi_{i_{t-1}} \underline{\Sigma}_{i,t}, \quad t > 0 \tag{4.10}$$

is obvious from direct calculation.

Remark: At this stage, an equivalent deterministic problem (Problem AE) will be defined with state $(\underline{\Sigma}_{i,t})_{i=0}^{L}$ for $t>0$ and state $\underline{\Sigma}_0$ at $t=0$. The system dynamics are then defined by equations (4.8) and (4.9).

Definition (Problem AE): For the system with matrix state $(\underline{\Sigma}_{i,t})_{i=0}^{L}$ for $t>0$ and $\underline{\Sigma}_0$ for $t=0$ with dynamical equations (4.8) and (4.9) and matrix control $\underline{G}_t$, minimize the equivalent deterministic cost over $(\underline{G}_t)_{t=0}^{T-1}$:

$$J_T^E = E_{\tilde{x},\underline{x}_0}\left[\sum_{t=0}^{T-1} \underline{x}_t^T Q \underline{x}_t + \underline{x}_t^T \underline{G}_t^T \underline{R}\underline{G}_t \underline{x}_t + \underline{x}_T^T Q \underline{x}_T | \underline{\Sigma}_0, \underline{\pi}_0\right]$$

$$= \sum_{t=0}^{T-1} \mathrm{tr}[\underline{\Sigma}_t (Q + \underline{G}_t^T \underline{R}\underline{G}_t)] + \mathrm{tr}[\underline{\Sigma}_T Q] \quad (4.11)$$

Note that since the expectation in equation (4.4) is over all structural trajectories $\tilde{x}$ and the initial $\underline{x}_0$ also,

$$J_T^E = J_T \quad (4.12)$$

The symbol J_T will be used exclusively in the future. The one-stage, or instantaneous, cost at time t is

$$J_T^t = \mathrm{tr}[\underline{\Sigma}_t (Q + \underline{G}_t^T \underline{R}\underline{G}_t)] \quad (4.13)$$

Problem AE is completely deterministic in the state $(\underline{\Sigma}_{i,t})_{i=0}^{L}$, $\underline{\Sigma}_0$ and control $\underline{G}_t$.

At this point, the minimization is decomposed into two parts using the Principle of Optimality [14]. The first minimization is over the interval $\{1,2,\ldots,T-1\}$, and for this the matrix minimum principle will be used. The resulting solution depends in general on the choice of $\underline{G}_0$ and on the initial conditions $\underline{\Sigma}_0$ and $\underline{\pi}_0$.

Let $V^*(\underline{G}_0)$ be the optimal cost resulting from the use of $\underline{G}_0$ and the optimal sequence $\underline{G}_1^*$, $\underline{G}_2^*, \ldots, \underline{G}_{T-1}^*$ for the interval $\{1,2,\ldots,T\}$. The second minimization is then over $\underline{G}_0$ of the cost

$$J_T = \mathrm{tr}[\underline{\Sigma}_0 (Q + \underline{G}_0^T \underline{R}\,\underline{G}_0)] + V^*(\underline{G}_0) \quad (4.14)$$

The Principle of Optimality states that these two minimizations result in the minimizing sequence $(\underline{G}_t^*)_{t=0}^{T-1}$ for Problem AE.

From [Athans,13], the Hamiltonian for the minimization over $\{1,2,\ldots,T-1\}$ is

$$H\left((\underline{\Sigma}_{i,t})_{i=0}^{L}, (\underline{S}_{j,t+1})_{j=0}^{L}, \underline{G}_t\right)$$
$$= \mathrm{tr} \sum_{i=0}^{L} \pi_{i_{t-1}} \underline{\Sigma}_{i,t} (Q + \underline{G}_t^T \underline{R}\,\underline{G}_t)$$
$$+ \mathrm{tr}\left[\sum_{j=0}^{L}\left(\frac{1}{\pi_{j_t}} \sum_{i=0}^{L} p_{ji}\pi_{i_{t-1}} (\underline{A}+\underline{B}_j \underline{G}_t)\underline{\Sigma}_{i,t} (\underline{A}+\underline{B}_j \underline{G}_t)^T\right) \underline{S}_{j,t+1}\right]$$
$$\text{for } t \in \{1,2,3,\ldots, T-1\} \quad (4.15)$$

where the costate matrix is $(\underline{S}_{j,t+1})_{j=0}^{L}$.

From the necessary condition for the costate,

$$\underline{S}_{i,t}^* = \frac{\partial H}{\partial \underline{\Sigma}_{i,t}}\bigg|_* \quad (4.16)$$

the propagation of $\underline{S}_{i,t}$ backward in time is derived.

$$\underline{S}_{i,t} = \pi_{i_{t-1}}\left\{ Q + \underline{G}_t^T \underline{R}\,\underline{G}_t + \sum_{j=0}^{L} p_{ji}\frac{1}{\pi_{j_t}} [\underline{A}^T \underline{S}_{j,t+1}\underline{A} + \underline{G}_t^T \underline{B}_j^T \underline{S}_{j,t+1}\underline{B}_j \underline{G}_t + \underline{A}^T \underline{S}_{j,t+1}\underline{B}_j \underline{G}_t + \underline{G}_t^T \underline{B}_t^T \underline{S}_{j,t+1}\underline{A}]\right\} \quad (4.17)$$

This equation is well-defined for any sequence $\{\underline{G}_t\}_{t=0}^{T-1}$ and $t>0$. The cost V of using this arbitrary sequence over the interval $\{1,2,\ldots,T\}$ is given by

$$V((\underline{G}_t)_{t=0}^{T-1}) = \mathrm{tr}\left\{\sum_{i=0}^{L} \underline{S}_{i,1}\underline{\Sigma}_{i,1}\right\} \quad (4.18)$$

Define

$$\underline{S}_0 \triangleq \sum_{i=0}^{L} (\underline{A}+\underline{B}_i \underline{G}_0)^T \underline{S}_{i,1} (\underline{A}+\underline{B}_i \underline{G}_0) + Q + \underline{G}_0^T \underline{R}\,\underline{G}_0 \quad (4.19)$$

Then the cost of a given sequence $(\underline{G}_t)_{t=0}^{T-1}$ of length T over the interval $\{0,1,\ldots,T\}$ is

$$J_T = \mathrm{tr}[\underline{\Sigma}_0 \underline{S}_0 (\underline{G}_0, \underline{G}_1, \ldots, \underline{G}_{T-1})] \quad (4.20)$$

From the Hamiltonian minimization necessary condition

$$\frac{\partial H}{\partial \underline{G}_t}\bigg|_* = \underline{0} \quad (4.21)$$

the following relation between $\underline{\Sigma}_{i,t}$, $\underline{S}_{j,t+1}$, and $\underline{G}_t$ is obtained.

$$\underline{0} = \underline{R}\,\underline{G}_t \sum_{i=0}^{L} \pi_{i_{t-1}} \underline{\Sigma}_{i,t} + \sum_{j=0}^{L} \frac{1}{\pi_{j_t}}\left[[\underline{B}_j^T \underline{S}_{j,t+1} \underline{B}_j \underline{G}_t + \underline{B}_j^T \underline{S}_{j,t+1}\underline{A}] \sum_{i=0}^{L} p_{ji}\, \pi_{i_{t-1}} \underline{\Sigma}_{i,t}\right] \quad (4.22)$$

<u>Remark</u>: At this point, a two-point boundary value problem has been defined with the constraint (4.22) relating equations (4.17) and (4.8). Equation (4.22) is not explicitly solvable for $\underline{G}_t$ because $\underline{\Sigma}_{i,t}$ cannot be factored out of the sum over j; thus it cannot be used as a substitution rule in the other two equations. At this time, the solution of $\underline{G}_t^*$ appears intractable. Thus, although necessary conditions for the existence of $\underline{G}_t^*$, the minimizing gain, have been established, they do not readily allow for the solution of $\underline{G}_t^*$, and certainly do not admit a closed-form expression.

5. Steady-State Non-Switching Gain Solutions

In this Section a modified version of Problem A is solved which yields a computational methodology for computing the optimal steady-state non-switching gain solutions. It will be established that the solution to this modified problem converges to the same limit as the problem in the last Section.

Definition 6: (Stability) $\underline{G}$ is a constant stabilizing gain if and only if the resulting system

$$\underline{x}_{t+1} = \underline{A}\,\underline{x}_t + \underline{B}_{k(t)}\,\underline{u}_t \tag{5.1}$$

is mean-square stable:

$$E[\underline{x}_t \underline{x}_t^T] \to 0 \quad \text{as } t \to \infty$$

Definition 7: (Cost-Stability) The system (5.1) is cost-stable if and only if

$$\sum_{t=0}^{\infty} \underline{x}_t^T \underline{Q}\,\underline{x}_t + \underline{u}_t^T \underline{R}\,\underline{u}_t < \infty$$

with probability one.

The infinite-time problem is defined as a minimization of

$$J = \lim_{T\to\infty} J_T \tag{5.2}$$

where J_T is the cost function for the corresponding finite-time problem. The sequence which solves the infinite-time versions of Problem AE is $(\underline{G}_t^*)_{t=0}^{\infty}$ when a solution exists. A solution will exist if there exists a sequence of gains for which the limit in equation (5.2) exists. This definition of the infinite-time problem is chosen rather than the definition requiring a minimization of the average cost per unit time

$$J_1 = \lim_{T\to\infty} \frac{1}{T} J_T \tag{5.3}$$

because there is a direct correlation between the boundedness of J_T over all T for a constant sequence of gains $\underline{G}$ and mean square stability of the system (5.1).

The concepts of stability, cost-stability, and existence of a steady-state solution are related by the following lemmas [12]:

Lemma 3: A constant sequence of gains $(\underline{G})_{t=0}^{\infty}$ is mean-square stabilizing if and only if there esists a bound $B<\infty$ such that $J_T < B \;\forall\; T$.

Lemma 4: Any sequence $(\underline{G}_t)_{t=0}^{\infty}$ cost-stabilizes (5.1) (with probability one) if and only if $J<\infty$.

The steady-state solution for Problem AE is defined as the limiting solution to equations (4.8) (4.17) and (4.22) at time t, first as $T\to\infty$ and then as $t\to\infty$, if this limit exists. The steady-state values for $\underline{B}$, $\underline{S}_i$, and $\underline{\Sigma}_i$, when they exist, satisfy the following equations:

$$\underline{\Sigma}_j = \frac{1}{\pi_j} \sum_{i=0}^{L} p_{ji}\,\pi_i\,(\underline{A}+\underline{B}_i\underline{G})\,\underline{\Sigma}_i\,(\underline{A}+\underline{B}_i\underline{G})^T \tag{5.4}$$

$$\underline{S}_i = \pi_i \left[\underline{Q} + \underline{G}^T\underline{R}\,\underline{G} + \sum_{j=0}^{L} p_{ji}\frac{1}{\pi_j} [\underline{A}^T\underline{S}_j\underline{A} + \underline{G}^T\underline{B}_j^T\underline{S}_j\underline{B}_j\underline{G} + \underline{A}^T\underline{S}_j\underline{B}_j\underline{G} + \underline{G}^T\underline{B}_j^T\underline{S}_j\underline{A}] \right] \tag{5.5}$$

$$\underline{0} = \underline{R}\,\underline{G}\sum_{i=0}^{L}\pi_i\,\underline{\Sigma}_i + \sum_{j=0}^{L}\frac{1}{\pi_j}\left[[\underline{B}_j^T\underline{S}_j\underline{B}_j\underline{G} + \underline{B}_j^T\underline{S}_j\underline{A}] \cdot \sum_{i=0}^{L} p_{ji}\,\pi_i\,\underline{\Sigma}_i \right] \tag{5.6}$$

which are the limit of these equations, given that the limiting solution $\underline{\Sigma}_{j,t}$ and $\underline{G}_t^*$ exist, where $\underline{\pi}$ satisfies

$$\underline{\pi} = \underline{P}\,\underline{\pi} \tag{5.7}$$

and

$$\lim_{t\to\infty} \underline{\pi}_t = \underline{\pi} \tag{5.8}$$

The following Theorem yields an explicit procedure for the calculation of the steady-state non-switching control law.

Theorem: Define the sequence $(\underline{G}_{ns})_{t=0}^{\infty}$ by the following equations:

$$\underline{G}_{ns_t}(T) = -[\underline{R} + \sum_{j=0}^{L}\pi_{j_t}\,\underline{B}_j^T\underline{S}'_{j,t+1}\underline{B}_j]^{-1} \cdot \sum_{j=0}^{L}\pi_{j_t}\,\underline{B}_j^T\underline{S}_{j,t+1}\,\underline{A} \tag{5.9}$$

for a given terminal time T and

$$\underline{G}_{ns_t} = \lim_{T\to\infty} \underline{G}_{ns_t}(T) \tag{5.10}$$

where

$$\underline{S}'_{k,t}(T) = \underline{Q} + \underline{G}^T_{ns_t}\underline{R}\,\underline{G}_{ns_t} \tag{5.11}$$

$$+ \sum_{j=0}^{L} p_{jk}(\underline{A}^T\underline{S}'_{j,t+1}\underline{A} + \underline{A}^T\underline{S}'_{j,t+1}\underline{B}_j\underline{G}_{ns_t} + \underline{G}^T_{ns_t}\underline{B}_j^T\underline{S}'_{j,t+1}\underline{A} + \underline{G}^T_{ns_t}\underline{B}_j^T\underline{S}'_{j,t+1}\underline{B}_j\underline{G}_{ns_t}) \quad \text{for } k\in I$$

$$\underline{S}'_{k,t}(T) = \underline{Q} \quad \text{for } k\in I \tag{5.12}$$

[The parameter (T) is suppressed on the right hand side of equations (5.9) and (5.11).]
Then the following statements are equivalent.

1) The gain sequence $(\underline{G}_{ns_t})_{t=0}^{\infty}$ cost-stabilizes $\underline{x}_{t+1} = \underline{A}\,\underline{x}_t + \underline{B}_{k(t)}\,\underline{u}_t$

2) $\left\| \lim_{T\to\infty} \sum_{k=0}^{L} \pi_{k,t-1}\underline{S}'_{k,t} \right\| < \infty$

3) A cost-stabilizing gain sequence exists.

4) The solution to Problem A, $(\underline{G}_t^*)_{t=0}^{\infty}$ is cost-stabilizing.

in addition, if

$$\underline{G}_{ns_t} = \underline{G}_{ns} \text{ for all } t \tag{5.13}$$

$\underline{G}^*$ (steady-state) exists then

$$\underline{G}_{ns} = \underline{G}^* \tag{5.14}$$

The proof can be found in Birdwell [12]. The derivation of equations (5.9),(5.11) and (5.12) can be found in Birdwell and Athan [11] and Birdwell [12]. A forthcoming paper will contain the complete theorem and proof. Equation (5.9) to (5.12) will hereafter be refered to as the solutions to Problem B, which is described in [12] and is omitted here due to lack of space. The results of this theorem give a direct computational procedure for calculating the optimal steady state gain $\underline{G}^*$ as the limit of gains $\underline{G}_{ns}$. There are some questions as to the possibility of limit cycles on the calculation of $\underline{G}_{ns}$. However, the theorem guarantees cost-stability using $\{\underline{G}_{ns_t}\}_{t=0}^{\infty}$ whenever the system is cost-stabilizable.

6. Robustness

The original problem (Problem A) can be formulated in such a way that the sequence $(\underline{G}_{ns_t})_{t=0}^{\infty}$ will cost-stabilize a set of linear systems with different actuator structures individually whenever such a stabilizing or robust gain exists.

Definition 8: A gain $\underline{G}$ is robust if

$$\underline{x}_{t+1} = (\underline{A} + \underline{B}_k \underline{G})\underline{x}_t \qquad (6.1)$$

is stable for all k. This is the same as requiring the matrix $(\underline{A}+\underline{B}_k\underline{G})$ to have eigenvalues inside the unit circle for all k.

Corollary 1: For the set of L+1 systems

$$\underline{x}_{t+1} = \underline{A}\,\underline{x}_t + \underline{B}_k \underline{u}_t \qquad (6.2)$$

with

$$\underline{P} = \underline{I} \qquad (6.3)$$

$$\pi_j = \frac{1}{L+1} \qquad (6.4)$$

if a robust gain exists, then $(\underline{G}_{ns_t})_{t=0}^{\infty}$ is a stabilizing sequence for (6.1) for each k, and if the gains $\underline{G}_{ns_t}$ (T) converge, then $\underline{G}_{ns}$ is a robust gain.

Discussion: With Corollary 1, a specific existence problem for robust linear gains is solved. Existence of a robust gain is made equivalent to the existence of a finite cost infinite-time solution to Problem B, which is readily computable from equations (5.9) and (5.11).

Consider the system whose transitions are shown in Figure 2. The configuration dynamics are modeled as being in any structural state with equal probability of occurance initially and remaining in that state forever; this model is illustrated graphically in Figure 2 below.

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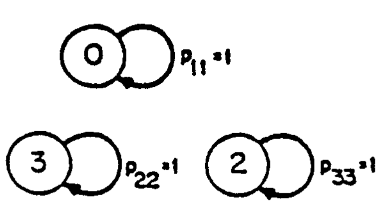

Figure 2: Markov transition probabilities for the Example.

The state dynamics are

$$\underline{x}_{t+1} = \underline{A}\,\underline{x}_t + \underline{B}_{k(t)}\underline{u}_t \qquad \underline{x}_t = [x_{1,t} \quad x_{2,t}]^T$$

$$k(t) \in \{0,1,2\}$$

The cost to be minimized is

$$J = E\left[\sum_{t=0}^{\infty} \underline{x}_t^T \underline{Q}\,\underline{x}_t + \underline{u}_t^T \underline{R}\,\underline{u}_t \,\middle|\, \underline{\pi}\right]$$

The matrices are given by

$$\underline{A} = \begin{bmatrix} 2.71828 & 0.0 \\ 0.0 & .3679 \end{bmatrix} \qquad \underline{P} = \begin{bmatrix} 1. & 0. & 0. \\ 0. & 1. & 0. \\ 0. & 0. & 1. \end{bmatrix}$$

$$\underline{B}_0 = \begin{bmatrix} 1.71828 & 1.71828 \\ -.63212 & .63212 \end{bmatrix}$$

$$\underline{B}_1 = \begin{bmatrix} 0.0 & 1.71828 \\ 0.0 & .63212 \end{bmatrix} \qquad \underline{B}_2 = \begin{bmatrix} 1.71828 & 0.0 \\ -.63212 & 0.0 \end{bmatrix}$$

For these matrices, equations (5.9) and (5.11) converge, giving the following results:

$$\underline{G}_{ns} = \begin{bmatrix} -1.089 & -.008413 \\ -1.028 & -.01444 \end{bmatrix}$$

$$\sum_{i=0}^{2} \pi_i \underline{S}_i' = \begin{bmatrix} 112.8 & 8.992 \\ 8.992 & 6.835 \end{bmatrix} \triangleq \underline{C}$$

A brief check will verify that this is indeed a robust gain. The Riccati solutions for this problem are

$$\underline{S}_0' = \begin{bmatrix} 109.8 & 9.030 \\ 9.030 & 6.821 \end{bmatrix} \qquad \underline{S}_1' = \begin{bmatrix} 114.3 & 6.285 \\ 6.285 & 6.836 \end{bmatrix}$$

$$\underline{S}_2' = \begin{bmatrix} 114.4 & 11.66 \\ 11.66 & 6.849 \end{bmatrix}$$

The non-switching solution converges for this system, and the three resulting configurations are stabilized. Therefore $\underline{G}_{ns}$ is a robust gain. Had the solution not converged, by Corollary 1, no robust gain would exist. The apriori expected cost (before the configuration state is known) is, given $\underline{x}$:

$$J = \underline{x}^T \underline{C}\,\underline{x}$$

7. Conclusion

In conclusion, the unifying concept of this report is: What constitutes a reliable control system, or a reliable design? A major connection was established in this research between the concepts of reliability and stabilizability. Iterative procedures were developed for the determination of whether or not a given linear system of the type considered in this report is reliable, with respect to both non-switching and switching gain controllers. A system design is reliable if and only if the set of coupled Riccati-like matrix difference equations for the switching gain solution converges. In addition, if the matrix difference equations converge for the non-switching gain solution, then the non-switching control law yields a robust system; if they diverge, no robust gain exists.

This paper is an overview of the results in Birdwell [12]. Two papers in preparation will contain the proofs of the results which are stated here.

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